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Quantum Group Approach to q-Special Functions

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Quantum groups are used to define q-special functions. The Casimir operators of a variation of $SU_q(2)$ and $E_q(2)$ are derived. The proposed q-associated Legendre and q-Bessel functions are the eigenfunctions of the Casimirs. The results differ from ordinary q-special functions, but this is expected since the q-generalization is not unique.

1. INTRODUCTION

Lie theory gives a natural setting for an algebraic interpretation of the special functions [1, 2]. Using the exponential mapping from the algebra to the corresponding group, one computes the matrix elements of group operators in specific irreducible representations and finds that these are typically expressible in terms of special functions. As an example, the group $SU(2)$ with diagonal subgroup isomorphic to $U(1)$, then the matrix elements of the irreducible representations of $SU(2)$ with respect to a $U(1)$-basis, can be expressed in terms of Jacobi polynomials, and the spherical functions are the associated Legendre polynomials. For quantum groups the situation is different. Only few quantum subgroups are available [3, 4].

Although q-special functions have been studied for several years [5, 6], the main line in the relation between quantum groups and q-special functions is a formal replacement of the parameters, say a by $[a]_q = (q^a - q^{-a})/(q - q^{-1})$ or the exponential mapping from the algebra g to the group G is replaced by q-exponentials from $U_q(g)$ into the completion of this algebra [7, 8].

In this paper we suggest a new approach to studying the relation between quantum groups and q-special functions by considering a quantum deforma-

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tion of the differential operator representations of the corresponding quantum algebra to get quantum deformed differential equations.

2. q-ASSOCIATED LEGENDRE

It is known that the associated Legendre polynomial $P_l^m(\theta)$ is derived from the $SU(2)$ algebra $J_\pm$, J_3 with

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_3 \tag{1}$$

The coordinate differential operator representation of this algebra is given by

$$\begin{aligned} J_\pm &= e^{\pm i\phi}(\pm\partial\theta + i \cot\theta\ \partial\phi) \\ J_3 &= -i\ \partial\phi \end{aligned} \tag{2}$$

This algebra has the Casimir operator

$$C = J_+ J_- + (J_3)^2 - J_3 \tag{3}$$

The eigenfunctions of the coordinate representation of the Casimir operator C are basis of the irreducible representation $D(2l)$. These are the spherical harmonics $Y_l^m(\theta, \phi) = e^{im\phi}P_l^m(\theta)$,

$$CY_l^m(\theta, \phi) = \lambda Y_l^m(\theta, \phi), \qquad \lambda = l(l + 1) \tag{4}$$

$P_l^m(\theta)$ is the associated Legendre polynomial. Now we define the following quantum deformed differential operators:

$$\begin{aligned} J_\pm &= e^{\pm\phi}(\pm\partial\theta - \cot\theta[-i\ \partial\phi]_q) \\ J_3 &= -i\ \partial\phi \end{aligned} \tag{5}$$

where $[-i\ \partial\phi]_q = (q^{-i\partial\phi} - q^{i\partial\phi})/(q - q^{-1})$ and $[-i\ \partial\phi]_q e^{im\phi} = [m]_q e^{im\phi}$.

One can prove that equations (5) are the differential representation of the quantum group $\tilde{S}U_q(2)$ [which differs a bit from the familiar $SU_q(2)$] with

$$\begin{aligned} [J_3, J_\pm] &= \pm J_\pm \\ [J_+, J_-] &= 2[J_3]_q \end{aligned} \tag{6}$$

$\tilde{S}U_q(2)$ has Casimir operator

$$C_q = J_+J_- + 2(q^{1/2} + q^{-1/2})\left[\frac{J_3}{2}\right]_q\left[\frac{J_3}{2} - \frac{1}{2}\right]_q \tag{7}$$

{a related form is $C'_q = J_-J_+ + 2(q^{1/2} + q^{-1/2})[\frac{1}{2}J_3]_q[\frac{1}{2}J_3 + \frac{1}{2}]_q$} and

$$C_q e^{im\phi}\ P_l^m(\cos\theta, q) = \lambda_q e^{im\phi}\ P_l^m(\cos\theta, q)$$

From equations (5) and (7) one gets the quantum deformed associated Legendre differential equation

$$-\frac{d^2 P_l^m(\cos\theta, q)}{d\theta^2} + \frac{dP_l^m(\cos\theta, q)}{d\theta} \cot\theta([m-1]_q - [m]_q)$$
$$+ P_l^m(\cos\theta, q)\{\mathrm{cosec}^2\theta[m]_q + \cot^2\theta[m]_q[m-1]_q + 2(q^{1/2} + q^{-1/2})$$
$$\times [m/2]_q[m/2 - 1/2]_q\} = \lambda_q P_l^m(\cos\theta, q) \qquad (8)$$

Without loss of generality we can take $\lambda_q = [l]_q[l+1]_q$. Put

$$A = [m]_q - [m-1]_q$$
$$B = [m]_q + [m]_q[m-1]_q$$
$$C = [m]_q + 2(q^{1/2} + q^{-1/2})\left[\frac{m}{2}\right]_q\left[\frac{m}{2} - \frac{1}{2}\right]_q - \lambda_q$$

and $x = \cos\theta$.

Equation (8) can be rewritten as

$$(x^4 - 2x^2 + 1)\frac{d^2P(x, q)}{dx^2} - (x - x^3)(1 + A)\frac{dP(x, q)}{dx}$$
$$- \{(B-C)x^2 + C\}P(x, q) = 0 \qquad (9)$$

This differential equation is an analytic differential equation except at $x = \pm 1$.

Let us assume that

$$P_l^m(x, q) = \sum_{n=0}^{\infty} a_n(q)x^n$$

Then one gets

$$a_2 = \frac{C}{2}a_0$$
$$a_3 = \frac{1 + A + C}{6}a_1 \qquad (10)$$
$$a_{n+2}(q) = \frac{[n(2n - 1 + A) + C]a_n(q) + [(2-n)(n - 2 + A) + B - C]a_{n-2}(q)}{(n+1)(n+2)},$$
$$n \geq 2.$$

3. q-BESSEL FUNCTIONS

The two-dimensional quantum Euclidean algebra $E_q(2)$ is defined by the following commutation relations:

$$[J_3, J_\pm] = \pm J_\pm \tag{11}$$
$$[J_+, J_-] = 0$$

It has the following quantum differential representation:

$$J_\pm = e^{\pm y}\left(\pm\partial_x - \frac{1}{x}[\partial_y]_q\right), \qquad J_3 = \partial_y \tag{12}$$

This quantum algebra $E_2(q)$ has the Casimir operator

$$C_q = J_+J_- \tag{13}$$

The eigenfunctions of C_q are given by $e^{my}J_m(x, q)$, where $J_m(x, q)$ is the q-Bessel function,

$$C_q e^{my}J_m(x, q) = \lambda_q e^{my}J_m(x, q) \tag{14}$$

From equations (12)–(14) we deduce the quantum differential equation of the Bessel function

$$-J''_m(x, q) - \frac{1}{x}J'_m(x, q) + \frac{([m]_q)^2}{x^2}J_m(x, q) = \lambda_q J_m(x, q)$$

$$J''_m(x, q) + \frac{1}{x}J'_m(x, q) + \left[\lambda_q - \frac{([m]_q)^2}{x^2}\right]J_m(x, q) = 0 \tag{15}$$

This differential equation has an ordinary singular point at $x = 0$. We consider

$$J_m(x, q) = \sum_{n=0}^{\infty} c_n(q)x^{n+r}$$

Then we get

$$c_n = \frac{\lambda_q}{r - (n + r)^2}c_{n-2}, \qquad n \geq 2$$
$$c_1 = 0$$
$$c_0 \neq 0$$

If $r = r_1 = [m]_q$, then

$$J_m^{(1)}(x, q) = x^{[m]_q}\sum_{n=0}^{\infty} c_n x^n$$
$$c_n = \frac{\lambda_q c_{n-2}}{[m]_q - (n + [m]_q)^2}, \qquad n \geq 2$$

and at $r = r_2 = -[m]_q$, one gets

$$J_m^{(2)}(x, q) = x^{-[m]_q} \sum_{n=0}^{\infty} c_n x^n$$

$$c_n = \frac{-\lambda_q c_{n-2}}{[m]_q + (n - [m]_q)^2}, \qquad n \geq 2$$

The general form of the q-Bessel function is

$$J_m(x, q) = AJ_m^{(1)}(x, q) + BJ_m^{(2)}(x, q)$$

such that $r_1 - r_2 \notin Z \cup \{0\}$ and Z is the set of positive integers.

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